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The Brauer Group and the Center of Generic Matrices

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INTRODUCTION

The primary objects of interest in this paper are the centers of the generic division algebras $UD(k, n, r)$ (e.g., [10, p. 91]). Let us recall the definition. Let k be a field and S the polynomial ring

$$k[x_{i,j,k} \mid 1 \leqslant i, j \leqslant n; 1 \leqslant k \leqslant r].$$

Let us consider the "generic matrices" X_k whose i, j entry is $x_{i,j,k}$. The ring of generic matrices $R(k, n, r)$ is the k algebra generated by the X_k's. It is a domain with a central localization $UD(k, n, r)$ which is a division algebra of degree n over its center $Z(k, n, r)$. This is the generic division algebra.

The basic underlying question is the structure of the extension $Z(k, n, r)/k$. For $n = 2$, 3, or 4, $Z(k, n, r)/k$ is known to be rational, that is, purely transcendental (see [6] for summary). For n a prime, $Z(k, n, r)/k$ is retract rational (see [22]). For all n, $Z(k, n, r)/k$ is obviously unirational.

In [2], Artin and Mumford showed that a certain unirational field, K, was not rational. Their proof can be viewed as follows. A nonsingular projective k variety V was exhibited such that $k(V)$, the function field of V, was K and such that V had nonzero Brauer group. But this says K cannot be rational.

The natural question is, then, whether there is such a k variety with $k(V) = Z(k, n, r)$. In [2], it was assumed that k was algebraically closed (making the result stronger). Here we do not assume this so the natural condition is to ask whether the Brauer group, $\mathrm{Br}(V)$, is unequal to $\mathrm{Br}(k)$. We in fact show that for any k, n, r, the Brauer group of any such V must be $\mathrm{Br}(k)$. As a step in our argument, we observe that if W is the Brauer Severi variety of $UD(k, n, r)$ defined over $K = Z(k, n, r)$, then $K(W)/k$ is rational.

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It should be noted that we first "algebraicize" the question by defining $\mathrm{Br}_u(K)$, that part of $\mathrm{Br}(K)$ unramified at every valuation ring of K/k, and showing that $\mathrm{Br}(V) \subseteq \mathrm{Br}_u(K)$. This also makes our argument actually independent of constructing a nonsingular model for $Z(k, n, r)$.

Section 3 deals with an independent question. In the center, C, of $R = R(k, n, r)$ there are elements, s, such that $R(1/s)/C(1/s)$ is Azumaya. More naturally, one looks at such s in the so-called trace ring, T, of R. One now asks about the Brauer group of $T(1/s)$ and whether it is generated by $RT(1/s)$. This obviously cannot be the case for all s, or k, but it is shown to be true in some natural cases.

To fix notation, recall that the Brauer group, $\mathrm{Br}(C)$, consists of equivalence classes of Azumaya algebras A/C. We denote by $[A]$ the equivalence class of A. If V is a scheme, then the Brauer group, $\mathrm{Br}(V)$, is the group of equivalence classes of locally Azumaya algebras. That is, the elements of $\mathrm{Br}(V)$ are equivalence classes of algebra sheaves over V, which are Azumaya when restricted to any affine open subset of V. The Brauer group is a covariant functor on the category of commutative rings and a contravariant functor on the category of schemes.

Let S/R be a (finite) cyclic Galois extension of commutative rings and let σ be a generator of the Galois group $\mathrm{Gal}(S/R)$. If $b \in R$ is a unit, then S/R, σ, and b define the cyclic algebra $\Delta(S/R, \sigma, b)$. This algebra is generated by S and z such that $zs = \sigma(s)z$ and $z^n = b$, where n is the degree of S/R (and the order of σ). Assume $R = F$ is a field of characteristic prime to n and that $\rho \in F$ is a primitive nth root of one. Then S is generated over F by v subject to the relations $v^n = a$ for some nonzero $a \in F$ and $\sigma(v) = \rho v$. In this case the algebra $\Delta(S/F, \sigma, b)$ is written $(a, b)_{n, F}$.

For any field F, the absolute Galois group of F will be the Galois group of F in its separable closure. If T is a commutative domain, $q(T)$ will denote the field of fractions of T. If $\varphi\colon R \to S$ is a ring homomorphism, then $\otimes_\varphi S$ will denote the tensor product of S viewed as an R module via φ. Finally, as a general rule, in situations where one map gives rise to a new one in an essentially unique way, the new map will be given the same symbol.

The research contained in this paper was motivated by a conversation this author had with W. Haboush. Some of the results here were also obtained by W. Haboush, though in a very different form and using very different methods. Unfortunately, as of this writing, Haboush's work is not yet written up. Also, this author would like to thank the referee for many helpful expositional comments, and for detecting the error in the original version of Theorem 3.1.

1. The Unramified Brauer Group

One goal of this paper is a result of the form: if K is the center of a generic division algebra over k, and V is a nonsingular projective k variety with function field K, then V will have Brauer group equal to that of k. We get our hands on $\mathrm{Br}(V)$ via a classical observation, as follows. Let T be any valuation ring with $q(T)=K$. Then T lies over an irreducible subvariety of V. Thus if $W \subseteq V$ is this subvariety, localization at W yields a subring of T. We now have the important but easy proposition which follows.

Proposition 1.1. *Let V be a nonsingular projective variety over k, and let $k(V)=K$ be the function field. If $T \subseteq K$ is a k algebra valuation ring with field of fractions K, then the image of* $\mathrm{Br}(V)$ *in* $\mathrm{Br}(K)$ *lies inside the image of* $\mathrm{Br}(T)$.

Proof. By p. 174 of [24], there is an irreducible closed subvariety of $W \subseteq V$ such that T lies over W. That is, if R, P is the local ring of V at W and M is the maximal ideal of T, then $R \subseteq T$ and $M \cap R = P$. In particular, the map $\mathrm{Br}(V) \to \mathrm{Br}(K)$ factors as

$$\mathrm{Br}(V) \to \mathrm{Br}(R) \to \mathrm{Br}(T) \to \mathrm{Br}(K)$$

and the result is proved. Q.E.D.

The maps mentioned above are actually all injective. This is well known for V and R, and was mentioned in [21] for T. In fact if X is a regular irreducible scheme with function field K, then $\mathrm{Br}(X) \to \mathrm{Br}(K)$ is injective [16, p. 145]. As for T, since no proof appears in print, we give one here.

Lemma 1.2. *Let T be a valuation ring with quotient field K. Then* $\mathrm{Br}(T) \to \mathrm{Br}(K)$ *is injective.*

Proof. The proof is almost word for word the same as the proof of Lemma 2.2 in [5, p. 136], though the later argument only ostensibly applies in the case T is a (rank one) discrete valuation domain. Here we will point out the one minor comment that generalizes the argument. So suppose A/T is Azumaya and $A \otimes_T K \cong \mathrm{End}_K(M)$, where M is a finite dimensional K vector space. View A as a subring of $\mathrm{End}_K(M)$ and choose $0 \neq m \in M$. Then Am is a finitely generated T submodule of M and so is a free T module. The rest of the proof now proceeds exactly as in [5].

Q.E.D.

With this result in hand, we will from now on consider the Brauer groups of objects like V, R, or T as subgroups of $\mathrm{Br}(K)$. What Proposition 1.1 says is that $\mathrm{Br}(V) \subseteq \cap\, \mathrm{Br}(T)$, the intersection being over all valuation rings $T \subseteq K$ with $q(T)=K$ and $k \subseteq T$.

We can now focus on the above intersection, and only consider it. That is, for any field $K \supseteq k$, we define $\mathrm{Br}_u(K)$ to be the intersection of $\mathrm{Br}(T)$ for all such valuation rings T. We now note two facts about $\mathrm{Br}_u(K)$.

PROPOSITION 1.3. *If $K \subseteq L$ are fields containing k, then the natural map $\mathrm{Br}(K) \to \mathrm{Br}(L)$ maps $\mathrm{Br}_u(K)$ into $\mathrm{Br}_u(L)$.*

Proof. Let $[A] \in \mathrm{Br}(K)$ be an element of $\mathrm{Br}_u(K)$. Suppose $T \subseteq L$ is a valuation ring with $q(T) = L$. Then $T \cap K = S$ is either K or a valuation ring in K with $q(S) = K$. Since $[A] \in \mathrm{Br}(S)$, $[A \otimes_K L] \in \mathrm{Br}(T)$ and we are done.

In [9], Hoobler generalized a result of Auslander–Goldman and showed that for a smooth affine domain R, $\mathrm{Br}(R) = \bigcap \mathrm{Br}(R_p)$, the intersection being over all height one primes of R. This immediately yields:

LEMMA 1.4. *If R is a smooth affine ring, and $q(R) = K$, then $\mathrm{Br}(R) \supseteq \mathrm{Br}_u(K)$.*

In [4], Auslander and Goldman showed that $\mathrm{Br}(k[x])/\mathrm{Br}(k)$ was either (0) or p primary if k had characteristic p. An easy argument shows that this is more generally true for $\mathrm{Br}(k[x_1,..., x_n])/\mathrm{Br}(k)$. This and Lemma 1.4 immediately show that: if $K = k(x_1,..., x_n)$, then;

LEMMA 1.5. *$\mathrm{Br}_u(K)/\mathrm{Br}(k)$ is (0) or p primary if k has characteristic p.*

To handle the remaining case, we will have to outline some known results about Brauer groups in characteristic p. For this discussion, let R be a commutative ring of characteristic p. If $a \in R$, set $R(a^{1/p})$ to be $R[y]/(y^p - a)$. Let v be the image of y in $R(a^{1/p})$. If $a, b \in R$, set $\{a, b\}$ to be the Azumaya R algebra generated by v, w subject to $v^p = a$, $w^p = b$, and $vw - wv = 1$ [12, p. 45]. We have identified $R(a^{1/p})$ with a subalgebra of $\{a, b\}$. Now define $\delta_a \colon R(a^{1/p}) \to R$ by

$$\delta_a(r_0 + r_1 v + \cdots + r_{p-1} v^{p-1}) = r_0^p + (p-1)!\, r_{p-1} + r_1^p a + \cdots + r_{p-1}^p a^{p-1}.$$

LEMMA 1.6. *Suppose $R(a^{1/p})$ has zero Picard group. Then $\{a, b\} \cong \{a, b'\}$ as R algebras if and only if $b - b' \in \delta_a(R(a^{1/p}))$.*

Proof. That $b - b' \in \delta_a(R(a^{1/p}))$ implies the isomorphism follows from [12, p. 45]. As for the converse, we begin by observing that $\{a, b\}$ and $\{a, b'\}$ are free as modules over $R(a^{1/p})$. Suppose $\varphi \colon \{a, b\} \to \{a, b'\}$ is an R isomorphism. The first step in our proof is the following argument which will show that we can choose φ to be the identity on $R(a^{1/p})$. We call the reader's attention to [17, Lemma 7.9, p. 88] which has a similar proof in a different context.

Set $A=\{a, b'\}$, and $S=R(a^{1/p})$. We will make the R module A into an $A\otimes_R S$ module in two ways. The first such module we will denote by the symbol A again. It is defined by setting $(a\otimes s)\cdot a'=aa's$. The second such module, gotten by twisting via φ, we denote by A'. It is defined by $(a\otimes s)\cdot a'=a\varphi(\varphi^{-1}(a')s)$. Here we are also considering $S\subseteq\{a, b\}$. Both A and A' are projective over S, and so are projective over $A\otimes_R S$ [5, p. 48]. Considering ranks we have that $\mathrm{End}_S(A)\cong A\otimes_R S\cong\mathrm{End}_S(A')$. By [5, p. 69], $A=A'\otimes_S E$, where E is a rank one projective S module. The assumption that S have zero Picard group says, exactly, that all such E are isomorphic to S. Thus $A\cong A'$.

Let $\psi: A\to A'$ be an $A\otimes_R S$ module isomorphism. In view of the definition of A and A', we can equivalently consider $\psi: A\to A$ with the property that $\psi(aa's)=a\varphi(\varphi^{-1}(\psi(a'))s)=a\psi(a')\,\varphi(s)$. Set $u=\psi(1)$. Since $\psi(v)=1$ for some $v\in A$, we have $1=\psi(v\cdot 1)=vu$. Also, $\psi(uv)=u=\psi(1)$ so $uv=1$. Altogether, u is invertible. We have $su=\psi(s\cdot 1)=\psi(1\cdot s)=u\varphi(s)$, so $\varphi(s)=u^{-1}su$ for all $s\in S$. If we change φ to $\varphi'(s)=u\varphi(s)\,u^{-1}$, we have that φ' is the identity on S. In other words, we may assume that φ is the identity on S as claimed.

Let v, w generate $\{a, b\}$ as above, and note that we have shown that we can assume $\varphi(v)=v$. Then $c=\varphi(w)-w'$ commutes with v and so must be in $R(a^{1/p})$. Using the results of [10, p. 187], we compute that $b=(w'+c)^p=(w')^p+\delta_a(c)=b'+\delta_a(c)$. This finishes the proof. Q.E.D.

We can now settle the outstanding case of Lemma 1.5.

PROPOSTION 1.7. *If K/k is rational, then* $\mathrm{Br}_u(K)=\mathrm{Br}(k)$.

Proof. By induction, we may assume $K=k(x)$. By Lemma 1.5, we may assume $[A]\in\mathrm{Br}_u(K)$ has exponent p, where p is the characteristic of k. By Lemma 1.4, [A] is the image of $[B]\in\mathrm{Br}(k[x])$. Since B has exponent p, it is split by $k^{1/p}[x^{1/p}]$ [12, p. 33]. Hence $[B]$ is a product in $\mathrm{Br}(k[x])$ of an element of the form $[\{x, g\}]$ and an element split by $k^{1/p}[x]$ [12, p. 45]. We will reach our conclusion by making use of the fact that $[A]$ is also in the image of $\mathrm{Br}(k[x^{-1}])$.

Set L to be the field $k^{1/p}$. The image of [B] in $\mathrm{Br}(L[x])$ is just $[\{x, g\}]$, and the image of $[A]$ is also in $\mathrm{Br}(L[x^{-1}])$. It follows that as Azumaya algebras over $L[x, x^{-1}]$, $[\{x, g\}]=[\{x^{-1}, g'\}]$, where $g'\in L[x^{-1}]$. An easy exercise shows that $\{x^{-1}, g'\}$ is isomorphic to $\{x, -x^{-2}g'\}$ over $L[x, x^{-1}]$. Hence as an algebra over $L[x, x^{-1}]$, $\{x, g+x^{-2}g'\}$ is split. Since $L[x, x^{-1}]$ is a principle ideal domain (p.i.d.), $\{x, g+x^{-2}g'\}\cong M_p(L[x, x^{-1}])\cong\{x, 0\}$. Finally, since $L[x^{1/p}, x^{-1/p}]$ is also p.i.d., Lemma 1.6 applies. Thus $g+x^{-2}g'\in\delta_x(L[x^{1/p}, x^{-1/p}])$. Another easy computation shows that this implies that $g\in\delta_x(L[x^{1/p}])$. In other words, $L[x]$ splits B.

Since $[B]$ is split by $k^{1/p}[x]$, it is split by $L'[x]$, where L'/k is finite, and $L' \subseteq k^{1/p}$. We will induct on the degree of L'/k. Choose $L'' \subseteq L'$ such that $L' = L''(a^{1/p})$, $a \in k$, and $L' \neq L''$. The image of $[B]$ in $\mathrm{Br}(L''[x])$ is $[\{a, f\}]$, for some $f \in L''[x]$. As above, the image of $[B]$ is also equal to $[\{a, f'\}]$, where $f' \in L''[x^{-1}]$. By the same argument, $f - f' \in \delta_a(L''(a^{1/p})[x, x^{-1}])$. Write $f = b + f''$, where $b \in L''$ and $f'' \in xL''[x]$. Yet another easy exercise (with δ_a) shows that $f'' \in \delta_a(L''(a^{1/p})[x])$ and so the image of $[B]$ is in $\mathrm{Br}(L'')$. Since $\mathrm{Br}(k) \to \mathrm{Br}(L'')$ is surjective, we can write $[B] = [B'][B'']$, where $[B'] \in \mathrm{Br}(k)$ and $[B'']$ is split by L''. As the image of $[B'']$ in $\mathrm{Br}(K)$ is also in $\mathrm{Br}_u(K)$, we can apply induction. Q.E.D

Before proceeding on with the main part of this paper, let us record an easy answer to a question arising in connection with previous work of this author. In [22], a retract rational extension K/k was defined to be an extension of fields such that $K = q(S)$ for an affine k domain S satisfying the following conditions. There are k algebra homomorphisms $i\colon S \to k[x_1, ..., x_n](1/s)$ and $j\colon k[x_1, ..., x_n](1/s) \to S$ such that $j \circ i$ is the identity on S.

Proposition 1.8. *Let S, K, i and j be as in the above definition. Also denote by i the extension $i\colon K \to k(x_1, ..., x_n)$.*

(1) *The induced map $i^*\colon \mathrm{Br}(K) \to \mathrm{Br}(k(x_1, ..., x_n)$ is an injection.*

(2) $\mathrm{Br}_u(K) = \mathrm{Br}(k)$.

Proof. Set $M = i(S - \{\emptyset\} \subseteq k[x_1, ..., x_n](1/s)$. Denote by T the localization of $k[x_1, ..., x_n](1/s)$ with respect to the multiplicatively closed set M. The maps i and j induce maps $i\colon K \to T$ and $j\colon T \to K$ such that $j \circ i$ is the identity on K. By functoriality, the induced map $\mathrm{Br}(K) \to \mathrm{Br}(T)$ is an injection. Since T is regular, $\mathrm{Br}(T) \to \mathrm{Br}(k(x_1, ..., x_n))$ is an injection. This proves (1), from which (2) immediately follows. Q.E.D.

2. The Center of Generic Matrices

Our goal in this section is to prove results about the Brauer group of the center of the generic division algebra. First, we must note a fact perhaps well known, but not anywhere in print as far as this author knows.

Theorem 2.1. *Let $R = R(k, n, r)$ be the ring of generic matrices and C the center of R. If $P \subseteq C$ is a prime ideal such that $R_P = RC_P$ is Azumaya over C_P, then C_P is a smooth algebra over k, and is the localization of an affine k algebra.*

Remark. This author is very grateful to W. Schelter for this fact and the proof which follows.

Proof. To show C_P is smooth, we use the infinitesimal criterion [15, p. 200]. Suppose S' is a k algebra with ideal $N \subseteq S'$ such that $N^2 = (0)$. Set $S = S'/N$. Assume $\varphi: C_P \to S$ is a k algebra map. To finish the proof, it suffices to show that φ lifts to a k algebra map $\varphi': C_P \to S'$.

R_P has center C_P, so $A = R_P \otimes_\varphi S$ is Azumaya with center S. By [12, p. 28], there is an Azumaya S' algebra A' such that $A'/NA' = A$. Call $X_1,..., X_r \in R$ the generic matrices generating R. φ induces a map $\varphi: R \to A$ and we can set $x_i = \varphi(X_i)$. Choose $x_i' \in A'$ to be preimages of the x_i. We define $\varphi': R \to A'$ by setting $\varphi'(X_i) = x_i'$. If $c \in C - P$, $\varphi'(c)$ is a preimage of $\varphi(c)$ and so is invertible. Thus φ' extends to R_P. The restriction of φ' to C_P is the desired map. To show the second property of C_P, note that there is a $c \in C - P$ such that $R(1/c)/C(1/c)$ is Azumaya. This implies that $R(1/c)$ is a finite module over $C(1/c)$. Also, $R(1/c)$ is an affine k algebra. By an exercise called the Artin–Tate lemma, $C(1/c)$ is an affine k algebra. Q.E.D.

Next we consider the field $K = Z(k, n, r)$ which is the center of $UD(k, n, r)$ and is the field of fractions of C above. Except in some special cases, it is not known whether K/k is rational. The next theorem, however, says that K becomes rational after extending by the function field of a Brauer Severi variety. So let $D = UD(k, n, r)$ and let V be the Brauer Severi variety defined by D over K. We consider the function field $K(V)$, which is the generic splitting field defined in [1] and studied further in [19].

THEOREM 2.2. *$K(V)/k$ is rational.*

Theorem 2.2 is actually stronger than we need. What would suffice for us is the quicker observation that by [21], $K(V)/k$ is stably rational. It seems worthwhile, however, to record this stronger fact, as it does not seem to appear anywhere.

To prove 2.2, we begin by recalling facts about Brauer factor sets (see [23]). Suppose L/F is a finite separable extension of degree n and $M \supseteq L$ is the Galois closure of L/F. Let $G = \mathrm{Gal}(M/F)$ and $H = \mathrm{Gal}(M/L)$. Let G act on $\{1,..., n\}$ as it acts on the cosets $\{gH \mid g \in G\}$, where 1 corresponds to the coset H. To any central simple F algebra with maximal subfield L there is an associated Brauer factor set $\{c(i, j, k) \mid 1 \leqslant i, j, k \leqslant n\} \subseteq M^*$, where the $c(i, j, k)$'s satisfy

(i) $\sigma(c(i, j, k)) = c(\sigma(i), \sigma(j), \sigma(k))$,

(ii) $c(i, j, k)\, c(i, k, m) = c(i, j, m)\, c(j, k, m)$

for all $\sigma \in G$ and all i, j, k, m between 1 and n. These Brauer factor sets can be used to describe Brauer Severi varieties as follows.

PROPOSITION 2.3. *Let L/F, G, H be as above and let A/F be a central simple F algebra with maximal subfield L, Brauer factor set $c(i,j,k)$, and Brauer Severi variety W. Form the purely transcendental extension $M' = M(y(1,i) \mid 2 \leqslant i \leqslant n)$. Let G act on M' via its usual action on M and $\sigma(y(1,i)) = (y(1,\sigma(i))/y(1,\sigma(1))\, c(1,\sigma(1),\sigma(i))$. Then this is a G action and the invariant field is isomorphic to $F(W)$.*

Proof. We will use the description of $F(W)$ contained in [19]. A is similar in the Brauer group to a crossed product $\Delta(M/F, G, d) = B$, where $d(\sigma,\tau) = c(1,\sigma(1),\sigma\tau(1))$ (e.g., [23, p. 213] where a difference in notation changes the formula slightly). $N = M \otimes_L A$ is a B module with M basis $\{x_i \mid 1 \leqslant i \leqslant n\}$, where $x_1 = 1 \otimes 1$ and $x_i = \sigma(x_i)$ for $\sigma \in B$ satisfying $\sigma(1) = i$. Write $B = \bigoplus_{\sigma\in G} Lu\,\sigma$ where $u_\sigma(a) = \sigma(a)\,u_\sigma$ and $u_\sigma u_\tau = d(\sigma,\tau)\,u_{\sigma\tau}$. Then $\sigma(x_i) = u_\sigma u_\tau(x_1) = d(\sigma,\tau)\,u_{\sigma\tau}(x_1) = c(i,j,k)\,x_k$, where $\sigma(1) = j$ and $\sigma(i) = \sigma\tau(1) = k$. The construction in [19] describes $F(W)$ as the invariant field of $M(x_i/x_1 \mid 2 \leqslant i \leqslant n)$ under the semi-linear action induced by the u_σ's. Identifying $y(1,i)$ with x_i/x_1 finishes the proof.

Getting back to generic matrices, we first reduce to a simpler case. Note that $UD(k,n,2)$ is canonically a subalgebra of $UD(k,n,r)$ for $r > 2$. This embedding satisfies $Z(k,n,2) \subseteq Z(k,n,r)$ and so $UD(k,n,2)\,Z(k,n,r) = UD(k,n,r)$. Also, it is known that $Z(k,n,r)/Z(k,n,2)$ is rational (e.g., [20, p. 197]). It follows that it suffices to prove 2.2 in the case $r = 2$.

The field $Z(k,n,2)$ is described in [6] as follows. Let S_n be the symmetric group with its action on $\{1,\ldots,n\}$, and let Q, P be the $\mathbb{Z}[S_n]$ modules with basis $\{x(i) \mid 1 \leqslant i \leqslant n\}$ and $\{y(i,j) \mid 1 \leqslant i,j \leqslant n\}$ such that $\sigma(x(i)) = x(\sigma(i))$ and $\sigma(y(i,j)) = y(\sigma(i),\sigma(j))$, respectively. There is an exact sequence of $\mathbb{Z}[S_n]$ modules

$$0 \to A \to P \to Q \to \mathbb{Z} \to 0,$$

where $y(i,j)$ maps to $x(i) - x(j)$ and $x(i)$ maps to 1. Let $L = k(Q \oplus A)$ be the field of fractions of the group algebra $k[Q \oplus A]$, and let S_n act on L in the natural way. $K = Z(k,n,2)$ is the fixed field. Note that we have written these $\mathbb{Z}[S_n]$ modules additively, but $Q \oplus A$ is identified with a subgroup of the *multiplicative* group of $k[Q \oplus A]$ and $k(Q \oplus A)$.

As an abelian group, A is generated by $c(i,j,k) = y(i,j) + y(j,k) = y(i,k)$. In fact, if $H \subseteq S_n$ is the subgroup fixing 1, then L^H is a maximal subfield of $UD(k,n,2)$ and $c(i,j,k)$ is the associated Brauer factor set. We form $L' = L(y(1,i)' \mid 2 \leqslant i \leqslant n)$ with S_n action as in 2.3. If we set $y(i,j)' = \sigma(y(1,k)')$, where $\sigma(1) = i$ and $\sigma(k) = j$ then the $y(i,j)'$ satisfy $y(i,j)'\,y(j,k)'/y(i,k)' = c(i,j,k)$ (check). Thus there is an isomorphism $L' \cong k(Q \oplus P)$, where S_n actions are preserved and the restriction to L is the identity. By 2.3, $K(V)$ is isomorphic to the invariant field of S_n on

$k(Q \oplus P)$. But if L'' is the S_n invariant field on $k(Q)$, then L''/k is rational. Since P is a permutation module, $K(V)/L''$ is rational by [14, p. 303]. This proves 2.2.

Since $K(V)/k$ is rational, we have that $\mathrm{Br}_u(K(V)) = \mathrm{Br}(k)$. Thus if $[A] \in \mathrm{Br}_u(K)$, then the image of $[A]$ in $\mathrm{Br}(K(V))$ lies in $\mathrm{Br}(k)$. Modifying $[A]$ by an element of $\mathrm{Br}(k)$, we can assume that $K(V)$ splits A. However, $\mathrm{Br}(K(V)/K)$ is exactly the subgroup of $\mathrm{Br}(K)$ generated by $[D] = [UD(k, n, r)]$ [19, p. 435]. We have

COROLLARY 2.4. *With notation as in* 2.2, $\mathrm{Br}_u(K)$ *is contained in the subgroup of* $\mathrm{Br}(K)$ *generated by* $\mathrm{Br}(k)$ *and* $[D]$.

The above corollary is a big step in the proof of the next theorem, one of the two main theorems in this paper. But before getting to this theorem, we must mention two facts we will need in the proof. Both facts are well known, but do not seem to appear anywhere conveniently. The first fact is the next lemma, which gives a relationship between regular rings and valuation rings.

LEMMA 2.5. *Let R be a regular local ring (and hence a domain). Set $q(R) = L$. Denote by M the maximal ideal of R. Then there is a valuation ring T with maximal ideal N such that $R \subseteq T \subseteq L$, $N \cap R = M$, and $T/N = R/M$.*

Proof. M is generated by an R sequence $x_1, \dots, x_d$ [15, p. 11]. Set $P = x_1 R$. Then R/P is a regular local ring of dimension one less. Now set $L' = q(R/P)$. If we set S to be the localization of R at P, then we can identify S/PS with L'. Also, S is a discrete valuation domain with maximal ideal PS. By induction, there is a valuation ring $T' \subseteq L'$ with maximal ideal N' such that $T' \supseteq R/P$, $N' \cap (R/P) = M/P$, and $R/M = T'/N'$. Set $T = \{a \in S \mid a + PS \in T'\}$ and $N = \{a \in S \mid a + PS \in N'\}$. Then, T, N are as claimed. Q.E.D.

The second fact we will require is a consequence of the following well-known description of the Brauer group of a complete discrete valued field. Let S be a complete discrete valuation domain with maximal ideal $P = vS$. Set $L = q(S)$. Denote by $L' \supseteq L$ the maximal unramified extension of L. Let $k = S/P$ and set k' to be separable closure of k. Then $G = \mathrm{Gal}(k'/k)$ can be identified with $\mathrm{Gal}(L'/L)$. The following exact sequence (e.g., [25, p. 186]) is the description mentioned above.

$$0 \to \mathrm{Br}(S) \to \mathrm{Br}(L'/L) \xrightarrow{\ \chi\ } \mathrm{Hom}_c(G, \mathbb{Q}/\mathbb{Z}) \to 0, \tag{2.6}$$

where Hom_c denotes the group of continuous homomorphisms.

Let us recall, in outline, the proof of (2.6). Denote by S' the integral closure of S in L'. S' is another discrete valuation domain with maximal ideal $P' = vS'$. If $(S')^*$, $(L')^*$ are the respective groups of units, there is an exact sequence of G modules:

$$0 \to (S')^* \to (L')^* \xrightarrow{\mu} \mathbb{Z} \to 0, \tag{2.7}$$

where μ is the (additive) valuation and G acts on $\mathbb{Z}$ trivially. Define $\eta: \mathbb{Z} \to (L')^*$ by setting $\eta(1) = v$. Then η is a right inverse for μ and (2.7) splits.

Consider the cohomology groups $H^n(G, \ )$ of the profinite group G. Apply the functor $H^n(G, \ )$ to (2.7). Since S' is local, $H^2(G, (S')^*) \cong \mathrm{Br}(S'/S)$. As S is complete, $\mathrm{Br}(S) \cong \mathrm{Br}(k)$ and $\mathrm{Br}(S'/S) = \mathrm{Br}(S)$. Of course, $H^2(G, (L')^*) \cong \mathrm{Br}(L'/L)$. Consider the exact sequence of G modules $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, where all G actions are trivial. As $H^n(G, \mathbb{Q}) = (0)$ for all $n \geqslant 1$, the long exact cohomology sequence yields an isomorphism $H^1(G, \mathbb{Q}/\mathbb{Z}) \cong H^2(G, \mathbb{Z})$. Since $\mathbb{Q}/\mathbb{Z}$ has trivial G action, $H^1(G, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_c(G, \mathbb{Q}/\mathbb{Z})$. This proves (2.6).

Our purpose in outlining the above argument is to allow us to compute the map χ from the sequence (2.6). Let N/L be an unramified, finite, cyclic extension of fields. Denote by $\mathrm{H} \subseteq G$ the Galois group $\mathrm{Gal}(L'/N)$. Set $n = [N: L]$ which is the order of G/H. Choose σ to be a fixed generator of $G/H = \mathrm{Gal}(N/L)$. Form the cyclic algebra $A = \Delta(N/L, \sigma, v)$. Let A correspond to $\alpha \in H^2(G/H, N^*)$. α is represented by a two cocycle $c: (G/H) \times (G/H) \to (L')^*$, where if $0 \leqslant i, j < n$, then $c(\sigma^i, \sigma^j) = 1$ for $i + j < n$, and $c(\sigma^i, \sigma^j) = v$ otherwise. Clearly α is in the image of $H^2(G/H, \eta(\mathbb{Z})) \cong H^2(G/H, \mathbb{Z}) = H^1(G/H, \mathrm{Q}/\mathbb{Z}) = \mathrm{Hom}(G/H, \mathrm{Q}/\mathbb{Z})$. An easy computation using c shows that α corresponds to $f: G/H \to Q/\mathbb{Z}$ such that $f(\sigma) = (1/n) + \mathbb{Z}$. All of these maps and isomorphisms commute with the inflation maps $H^2(G/H, N^*) \to H^2(G, (L')^*)$, $\mathrm{Hom}(G/H, Q/\mathbb{Z}) \to \mathrm{Hom}_c(G, Q/\mathbb{Z})$, etc. Altogether, $\chi([A])$ is the element $f': G \to Q/\mathbb{Z}$ such that $f'(H) = 0 + \mathbb{Z}$ and $f'(\sigma') = (1/n) + \mathbb{Z}$, where $\sigma' H = \sigma$.

We are finally prepared to prove the second fact we need.

LEMMA 2.8. *Let S be a not necessarily complete discrete valuation domain with completion S''. Set $q(S) = L$, and $q(S'') = L''$. Suppose that N/L is a cyclic, unramified, extension of fields of degree n such that, in addition, $N \otimes_L L''$ is a field. Denote by v a prime of S, and form $A = \Delta(N/L, \sigma, v)$. Then $[A]^m$ is in $\mathrm{Br}(S)$ if and only if n divides m.*

Proof. If n divides m, $[A]^m = 1$ so $[A]^m \in \mathrm{Br}(S)$ trivially. Conversely, if n does not divide m, it is enough to show $[A \otimes_L L'']^m \notin \mathrm{Br}(S'')$. That is, we may assume S is complete. Our computation of $\chi([A])$ shows that it has order n. The lemma follows from (2.6). Q.E.D.

We are finally ready for the promised theorem.

THEOREM 2.9. (1) $\mathrm{Br}_u(Z(k,n,r)) = \mathrm{Br}(k)$.

(2) *If V is a nonsingular projective k-variety with $k(V) = Z(k,n,r)$, then* $\mathrm{Br}(V) = \mathrm{Br}(k)$.

Proof. Of course, (2) is implied by (1). Let D, K be as above. Since $\mathrm{Br}(k) \subseteq \mathrm{Br}_u(Z(k,n,r))$, it suffices to show that:

LEMMA 2.10. *If $[D]^m \in \mathrm{Br}_u(Z(k,n,r))$, then $[D]^m = [1]$.*

Proof. Recall that $UD(k,n,r)$ is the central localization of $R = R(k,n,r)$. Denote the generic matrices generating R by $X, Y, Z_3, ..., Z_r$.

Let S be the localization of the polynomial ring $k[u,v]$ at the prime, P, generated by v. Then $S \subseteq k(u,v)$ and S naturally contains $k(u)$. Choose $L/k(u)$ to be a cyclic extension of fields of degree n. L is generated, over $k(u)$, by an element x such that x is integral over $k[u]$. Set $L' = L \otimes_{k(u)} k(u,v)$. Let σ denote a fixed generator of the Galois group $\mathrm{Gal}(L'/k(u,v))$. Form the cyclic algebra $A = \Delta(L'/k(u,v), \sigma, v)$. It is easy to see that 2.8 applies to A with respect to the discrete valuation ring S. Denote by $y \in A$ the canonical element such that $y^n = v$ and $yfy^{-1} = \sigma(f)$ for $f \in L'$. Define a k algebra homomorphism $\varphi\colon R \to A$ by setting $\varphi(X) = x$, $\varphi(Y) = y$, and $\varphi(Z_i) = 0$ for $i \geqslant 3$. Then $\varphi(R)\,k(u,v) = A$. Also, x, y are integral over $k[u,v]$ so $\varphi(R)$ is an order over $k[u,v]$. If C is the center of R, $\varphi(C)$ centralizes x and y so $\varphi(C) \subseteq k[u,v]$.

Let Q be the kernel of φ intersect C. We claim that $R_Q = RC_Q$ is Azumaya over C_Q. In order to prove this, it is most convenient to use the language of central polynomials (e.g., [20, p. 24]). Let q be a homogeneous central polynomial for $n \times n$ matrices over k. Since $\varphi(R)\,k(u,v) = A$, q is not an identity on $\varphi(R)$. Thus there is a $c \in C$ such that $c = q(r_1, ..., r_m)$ for $r_i \in R$ and $\varphi(c) \neq 0$. By the Artin–Processi theorem (e.g., [20, p. 70]), $R(1/c)/C(1/c)$ is Azumaya. As $c \notin Q$, R_Q is Azumaya over C_Q.

By 2.1, C_Q is a regular local ring. By 2.5, there is a valuation ring T with maximal ideal M such that: $C_Q \subseteq T \subseteq Z(k,n,r)$, $M \cap C_Q = QC_Q$, and $T/M = C_Q/QC_Q$. Hence there is an induced map $q\colon T \to k(u,v)$. Let $U = q^{-1}(S)$. As S is a valuation ring, U is a valuation ring. Let q' denote the restriction of q to U. Altogether we have the diagram:

$$\begin{array}{ccc} U & \subseteq & T \\ {\scriptstyle q'}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ S & \subseteq & k(u,v). \end{array} \tag{2.11}$$

Note that $RT = R_Q T$ is Azumaya over T. If q'^*, q^* denote the induced maps on Brauer groups, then $q^*([RT]) = [A]$. Let $D = UD(k, n, r)$ and assume that $[D]^m \in \mathrm{Br}_u(\mathbb{Z}(k, n, r))$. Then $[D]^m$ is the image of some $\alpha \in \mathrm{Br}(U)$. Since $\mathrm{Br}(T) \to \mathrm{Br}(\mathbb{Z}(k, n, r))$ is injective, α must also be the preimage of $[RT]^m \in \mathrm{Br}(T)$. From (2.11), $q'^*(\alpha)$ must be the preimage of $[A]^m \in \mathrm{Br}(k(u, v))$. Using 2.8, we know that n divides m and so that $[D]^m = 1$. Q.E.D.

3. The Brauer Group of a Subring

In this section we will construct a smooth subring $T \subseteq Z(k, n, r)$ such that $\mathrm{Br}(T)$, viewed as in $\mathrm{Br}(Z(k, n, r))$, is generated by $[UD(k, n, r)]$. To this end, we start with some results about the Brauer group of localized polynomial rings. In this section k will always be an algebraically closed field of characteristic zero. The polynomial ring $k[x_1,..., x_n]$ is a *UFD*, so we can talk about prime elements.

Let $s \in k[x_1,..., x_n]$ be a prime. If S is the localization of $k[x_1,..., x_n]$ at the ideal generated by s, then S is a discrete valuation ring with $q(S) = k(x_1,..., x_n) = (\text{say})\ K$. Hence S induces a map $\chi_s\colon \mathrm{Br}(K) \to \mathrm{Hom}_c(G, \mathbb{Q}/\mathbb{Z})$, where G is the absolute Galois group of L, the residue field of S (2.6). Since L contains all roots of one, the n torsion subgroup of $\mathrm{Hom}_c(G, \mathbb{Q}/\mathbb{Z})$ can be identified with $L^*/(L^*)^n$. In the arguments to follow we will often make this identification.

Lemma 3.1. *Suppose $s \in k[x_1,..., x_n]$ is irreducible and has the form $s = tx_n + q$, where $t, q \in k[x_1,..., x_{n-1}]$. If $[A] \in \mathrm{Br}(k[x_1,..., x_n]\,(1/s))$ has prime exponent p, and $\chi_s([A]) \neq 1$, then $\chi_s([A]) = a(L^*)^p$, where $a \in k[x_1,..., x_{n-1}]$ is such that all primes dividing a also divide t.*

Proof. Under the identification mentioned above, let $\chi_s([A]) = a(L^*)^p$. We can choose "a" such that $a \in k[x_1,..., x_{n-1}]$ and all primes dividing "a" appear to a prime to p power. Define $[B] \in \mathrm{Br}(K)$ by the relation $[A] = [(a, s)_{p,K}]\,[B]$. Using the calculation preceding Lemma 2.8, we have that $\chi_s([B]) = 1$.

Let r be a prime dividing a, and assume that r does not divide t. Since x_n does not appear in r, $k[x_1,..., x_n]/(r)$ is a polynomial ring in x_n over $k[x_1,..., x_{n-1}]/(r)$. Here (r) is the ideal generated by r in the appropriate ring. As r does not divide t, s has degree exactly one in x_n modulo r.

We next consider the ramification of $[B]$. Let $w \in k[x_1,..., x_n]$ be a prime not dividing "as", and let T be the localization of $k[x_1,..., x_n]$ at the prime ideal (w). Since $s \notin (w)$, $[A] \in \mathrm{Br}(T)$. As "as" is a unit in T, $[(a, s)] \in \mathrm{Br}(T)$. Hence $[B] \in \mathrm{Br}(T)$. Since $\chi_s([B]) = 1$, $[B] \in \mathrm{Br}(T)$ in the case $w = s$ also. In particular, if w is any prime such that

$(w) \cap k[x_1, ..., x_{n-1}] = (0)$, and T is the above localization, then $[B] \in \mathrm{Br}(T)$. By Hoobler's result [9], $[B]$ is in the image of $\mathrm{Br}(k(x_1, ..., x_{n-1}))$. Next we compute $\chi_r([B])$. Since $s \notin (r)$, $\chi_r([A)] = 1$. If m is the highest power of r dividing a, $\chi_r([a, s)])^{-1} = \chi_r([(s, a)]) = (s')^m (L'^*)^p$, where s' is the image of s modulo r and $L' = q(k[x_1, ..., x_n]/(r))$. Thus $\chi_r([B]) = (s')^m (L'^*)^p$. On the other hand, as $[B]$ is in the image of $\mathrm{Br}(k(x_1, ..., x_{n-1}))$, $\chi_r([B])$ must have the form $b(L'^*)^p$, where b is in the image of $k[x_1, ..., x_{n-1}]$. This contradicts the fact that s' has degree one in x_n, and this contradiction proves the lemma.

Q.E.D.

We will now use 3.1 to show that for a particular s, $\mathrm{Br}(k[x_1, ..., x_n] (1/s)) = (0)$. Let $n = m^2$ and relabel the x_i's as x_{ij} for $1 \leqslant i, j \leqslant m$. Set s to be in the determinant of the matrix with (i, j) entry x_{ij}.

PROPOSITION 3.2. *With this definition of* s,

$$\mathrm{Br}(k[x_{ij} \mid 1 \leqslant i, j \leqslant m] \ (1/s)) = (0).$$

Proof. Assume that there is an $[A] \in \mathrm{Br}(k[x_{ij} \mid 1 \leqslant i, j \leqslant m] \ (1/s)$ of prime exponent p. If w is any prime distinct from s, then $\chi_w([A]) = 1$. If $\chi_s([A]) = 1$, then $[A] \in \mathrm{Br}(k[x_{ij} \mid 1 \leqslant i, j \leqslant m])$. Since the Brauer group of a polynomial ring over k is (0), we must have $\chi_s([A]) \neq 1$. Write $s = x_{11}s' + s''$, where s' is the determinant associated with the (1, 1) minor. By 3.1, $\chi_s([A]) = t(L^*)^p$, where $L = q(k[x_{ij} \mid 1 \leqslant i, j \leqslant m]/(s))$ and t is a prime to p power of s'. By altering $[A]$, we may assume $t = s'$. Write $[A] = [(s', s)_p] [B]$ as in 3.1. Set R to be the polynomial ring $k[x_{ij} \mid 1 \leqslant i, j \leqslant m;\ (i, j) \neq (1, 1)]$, $K = q(R)$, and $L' = q(R/(s'))$. Arguing as in 3.1, $[B]$ is in the image of $\mathrm{Br}(K)$, and $\chi_{s'}([B]) = s''(L')^p$. Also, $\chi_r([B]) = 1$ for any prime r of R unequal to s'. That is, $[B] \in \mathrm{Br}(R(1/s'))$. Now s' is itself a determinant and we can write $s' = x_{22}s^* + \hat{s}$. Applying 3.1 again we have that $\chi_{s'}([B])$ is some power of $s^*(L'^*)^p$. However, x_{12} does not appear in s' and modulo s', s'' has degree one in x_{12}. On the other hand, s^* has degree zero in x_{12}. This contradiction proves the result.

Q.E.D.

We can now turn to certain subrings of $Z(k, n, r)$. Recall that $R = R(k, n, r)$ is defined as a subring of $M_n(S)$, where $S = k[x_{ijl} \mid 1 \leqslant i, j \leqslant n;\ 1 \leqslant l \leqslant r]$. Set C to be the center of R and set T to be a ring generated over C by the traces of all elements of R. T is the so-called trace ring. S has a natural action by $PGL_n(k)$ under which T is the fixed ring [7, 18]. Set $A = RT$. Now consider the special case $r = n^2$. Set $s \in S$ to be the determinant of the $n^2 \times n^2$ matrix whose kth column consists of the x_{ijk}'s in lexicographical order. A calculation shows that s is fixed by $PGL_n(k)$ and so s is in T.

THEOREM 3.3. *The Brauer group* $\mathrm{Br}(T(1/s))$ *is cyclic of order* n *generated by* $[A(1/s)]$.

Proof. The generic matrices X_l form a basis of $M_n(S(1/s))$. Thus $RS(1/s) = M_n(S(1/s))$. Artin's theorem (e.g., [20, p. 70]). now shows that $RT(1/s)/T(1/s)$ is Azumaya.

To finish the proof, we consider the diagram:

$$\begin{array}{ccc} \mathrm{Br}(S(1/s)) & \longrightarrow & \mathrm{Br}(k(x_{ijl})) \\ \uparrow & & \uparrow \\ \mathrm{Br}(T(1/s)) & \longrightarrow & \mathrm{Br}(Z(k, n, n^2)). \end{array}$$

As $AZ(k, n, n^2) = UD(k, n, n^2) = D$, the image of $\mathrm{Br}(T(1/s))$ contains $[D]$. By [13], $k(x_{ijl})$ is a generic splitting field for D. Hence $[D]$ generates the kernel of the vertical map on the right. As $S(1/s)$ is smooth, the upper horizontal map is injective.

We next observe that $T(1/s)$ is smooth. To begin with, T is an affine k algebra [20, p. 209]). Let $P \subseteq T(1/s)$ be a prime ideal. Because $A(1/s)$ is Azumaya, $P = PA(1/s) \cap T(1/s)$.

If $P' = P \cap T$ then $A_{P'}/T_{P'}$ is Azumaya. In other words, $P'A$ is in what is called $\mathrm{Spec}_n(A)$ (e.g., [20, p. 75]). Thus $P'A$ cannot contain all valuations of central polynomials on R, and so $P'A \cap R$ is in $\mathrm{Spec}_n(R)$. All together, $Q = P \cap C$ is in $\mathrm{Spec}_n(C)$, and so R_Q/C_Q is Azumaya. An Azumaya algebra is closed under the trace map so $C_Q \supseteq T$. It follows that $C_Q = T_{P'} = T(1/s)_P$. By 2.1, $T(1/s)_P$ is smooth. Since smoothness is a local property (e.g., [16, p. 31]), $T(1/s)$ is smooth.

From the above paragraph we conclude that the lower horizontal map in diagram (3.4) is also injective. But by 3.2, $\mathrm{Br}(S(1/s)) = (0)$. Thus the image of $\mathrm{Br}(T(1/s))$ is also contained in the group generated by $[D]$, and we are done. Q.E.D.

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